QUIVER GRASSMANNIANS ASSOCIATED WITH STRING MODULES

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Abstract. We provide a technique to compute the Euler–Poincaré characteristic of a class of projective varieties called quiver Grassmannians. This technique applies to quiver Grassmannians associated with "orientable string modules". As an application we explicitly compute the Euler–Poincaré characteristic of quiver Grassmannians associated with indecomposable preprojective, preinjective and regular homogeneous representations of an affine quiver of type $\tilde{A}_{p,1}$. For $p = 1$, this approach provides another proof of a result due to P. Caldero and A. Zelevinsky in [9].

keywords. cluster algebras, cluster character, quiver Grassmannians, Euler characteristic, string modules.

1. Introduction and main results

In this paper we provide a technique to compute the Euler–Poincaré characteristic of some complex projective varieties called *quiver Grassmannians*. In the last few years many authors have shown the importance of such projective varieties and of their Euler–Poincaré characteristic in the theory of cluster algebras (see [5], [7], [6], [14]), introduced and studied by S. Fomin and A. Zelevinsky ([16], [17], [18]).

Given a quiver Q and a Q–representation M, the quiver Grassmannian $Gr_{\mathbf{e}}(M)$ is the set of all sub–representations of M of a fixed dimension vector $\mathbf{e}$ (see section 1.1). This is a complex projective variety and our aim is to compute its Euler–Poincaré characterisitc $\chi_{\mathbf{e}}(M)$. Our main result (theorem 1) says that under some technical hypotheses on M, there is an algebraic action of the one–dimensional torus $T = \mathbb{C}^*$ on $Gr_{\mathbf{e}}(M)$. It is well–known (see section 2) that if a complex projective variety is endowed with an algebraic action of a complex torus with finitely many fixed points, then its Euler–Poincaré characteristic equals the number of fixed points of this action and, in particular, it is positive. In general it is not true that the Euler–Poincaré characteristic of a quiver Grassmannian is positive (see [14, example 3.6]) but it is proved in [23] for quiver Grassmannians associated with rigid representations of acyclic quivers, as conjectured in [16]. The fixed points of the action of T on $Gr_{\mathbf{e}}(M)$ are the "coordinate" subrepresentations of M of dimension vector $\mathbf{e}$ (section 1.2). As a combinatorial tool to count them, we consider the coefficient quiver introduced by Ringel (see section 1.3) and we notice that its successor closed subquivers are in bijection with coordinate subrepresentations of M (proposition 1).

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We prove that "orientable string modules" (see definition 1.1) satisfy the hypotheses of theorem 1. Such a class of Q–representations includes (up to "right–equivalence") all the representations of the affine quiver of type $\tilde{A}_{p,1}$ and most of the representations of the affine quiver of type $\tilde{A}_{p,q}$.

As an application we explicitly compute $\chi_{\mathbf{e}}(M)$ when M is an indecomposable preprojective, preinjective and regular homogeneous representation of the affine quiver of type $\tilde{A}_{p,1}$. We hence find another proof of results of [9] for $p = 1$, and of [20] and [21] for $p = 2$. Such computations can be used to have an explicit description of the bases of cluster algebras of type $\tilde{A}_{p,q}$ found in [21] and [15] and for further studies of such cluster algebras [10]. In addition it would be interesting to compare our computations with results of [22] where the authors compute the Laurent expansion of cluster variables of cluster algebras arising from surfaces. In particular this gives a technique to compute the Euler–Poincaré characteristic of quiver Grassmannians associated with rigid representations of quivers associated with triangulations of surfaces with marked points. This family includes quivers of type $\tilde{A}_{p,q}$ where our technique applies. In type A one can compare our results with results of [1].

To conclude the introduction we remark that having a torus action on a smooth projective variety X gives rise to a cellular decomposition of X ([4], [11]). It is known that if M is a rigid Q–representation (i.e. without self–extensions) then $Gr_{\mathbf{e}}(M)$ is smooth [8]. In particular if M is a rigid Q–representation satisfying hypothesis of theorem 1 then $Gr_{\mathbf{e}}(M)$ has a cellular decomposition. This approach is used in [10].

The paper is organized as follows: in section 1.1 we recall some basic facts about quivers and quiver Grassmannians; in section 1.2 we state our main result; in section 1.3 we introduce the coefficient quiver of a Q–representation and we show how to use it as a combinatorial tool to apply the main result; in section 1.4 we introduce orientable string modules and we prove that they satisfy the hypotheses of our main theorem; in section 1.5 we give an explicit application for quivers of type $\tilde{A}_{p,1}$. All the remaining sections are devoted to proofs.

1.1. **Quiver Grassmannians.** We recall the definition of quiver Grassmannians. Given a quiver $Q = (Q_0, Q_1)$, i.e. an oriented graph with vertex set $Q_0 = \{1, \cdots, n\}$ and arrow set Q_1, a *Q–representation* M consists of a collection of complex vector spaces $\{M(i),\, i \in Q_0\}$ and a collection of linear maps $\{M(a) : M(j) \to M(i) \mid a : j \to i \in Q_1\}$.

Example 1. *The first column of table 1 shows some examples of quivers Q and the second one shows an example of a Q–representation M. We denote by k the field of complex numbers. In the last two rows we use the notation $E_{i,j}$ to denote the linear operator on k^4 which sends the j–th basis vector to the i–th one and fixes all the others.*

A *subrepresentation* N of M consists of a collection of vector subspaces $N(i)$ of $M(i)$, $i \in Q_0$, such that $M(a)N(j) \subset N(i)$ for every arrow $a : j \to i$ of Q. For example the Q–representation M shown in the first line of table 1 does *not* admit the Q–representation ($k \longrightarrow 0$) as its subpresentation (because the map $M(a)$ has one–dimensional image) but admits ($0 \longrightarrow k$).

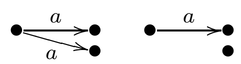
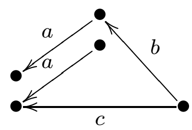
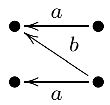
	Q	M	$\tilde{Q}(M)$
1	$1 \xrightarrow{a} 2$	$k \xrightarrow{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]} k^2 \simeq k \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} k^2$	
2	$1 \underset{a}{\overset{b}{\leftleftarrows}} 2$	$k \underset{1}{\overset{1}{\leftleftarrows}} k$	$\bullet \underset{a}{\overset{b}{\leftleftarrows}} \bullet$
3	$\begin{tikzcd} & 2 \arrow[dl, "a"'] & \\ 1 & & 3 \arrow[ul, "b"'] \arrow[ll, "c"] \end{tikzcd}$	$\begin{tikzcd} & k^2 \arrow[dl, "Id_2"'] & \\ k^2 & & k \arrow[ul, "\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]"'] \arrow[ll, "\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]"] \end{tikzcd}$	
4	$1 \underset{a}{\overset{b}{\leftleftarrows}} 2$	$k^2 \underset{Id}{\overset{J_2(0)}{\leftleftarrows}} k^2$	
5	$a \circlearrowleft 1 \circlearrowright b$	$E_{21}+E_{43} \circlearrowleft k^4 \circlearrowright E_{32}$	$\bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet \xrightarrow{a} \bullet$
6	$a \circlearrowleft 1 \circlearrowright b$	$E_{21}+E_{34} \circlearrowleft k^4 \circlearrowright E_{32}$	$\bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet \xleftarrow{a} \bullet$

TABLE 1. Some Q–representations and their coefficient–quiver. In the fourth row, we denote by $J_2(0)$ the 2×2 nilpotent Jordan block. In the last two rows E_{ij} denotes the 4×4 elementary matrix with 1 in the ij–component and zero elsewhere.

The *dimension vector* of M is the vector $\mathbf{dim}(M) := (dim_{\mathbb{C}}(M(i)) : i \in Q_0)$ where $dim_{\mathbb{C}}(M(i))$ denotes the complex dimension of the vector space $M(i)$. For example in table 1 the dimension vector of M is respectively, from above to below, $(1, 2)$, $(1, 1)$, $(2, 2, 1)$, $(2, 2)$, (4), (4).

The *path algebra* kQ of Q is the complex vector space with as basis the *paths* of Q (i.e. concatenations of arrows) endowed with the multiplication given by the juxtaposition of paths. It is known (see e.g. [3]) that the category of Q–representations is equivalent to the category of kQ–modules. In particular every Q–representation can be seen as a kQ–module and viceversa every kQ–module has a natural structure of Q–representation.

Finally, the *quiver Grassmannian* $Gr_{\mathbf{e}}(M)$ of M of dimension $\mathbf{e} = (e_i : i \in Q_0)$ is defined as the set of all the subrepresentations of M of dimension vector $\mathbf{e}$, that is,

$$Gr_{\mathbf{e}}(M) := \{N \subset M : \mathbf{dim}(N) = \mathbf{e}\}.$$

Example 2. *For the Q–representations M shown in lines 1 and 2 of table 1 the quiver Grassmannian $Gr_{(1,1)}(M)$ is a point. If M is the Q–representation of line 3, $Gr_{(1,1,1)}(M)$ is the empty set. Let M be the Q–representation shown in line 4. Here $J_2(0) = E_{12} = \left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]$ is the 2×2 nilpotent Jordan block which sends the second basis vector to the first one.*

We consider the set $Gr_{(1,1)}(M)$ of subrepresentations of M of dimension vector $(1,1)$. This consists of lines in k^2 spanned by non–zero vectors $v = (\lambda, \mu)^t \in k^2$ such that v and $J_2(0)v$ are linearly dependent. In other words a line spanned by v is in $Gr_{(1,1)}(M)$ if and only if $\det \begin{bmatrix} \lambda & \mu \\ \mu & 0 \end{bmatrix} = -\mu^2 = 0$. *Then $Gr_{(1,1)}(M)$ is a point which is actually not reduced, indeed the tangent space at this point has dimension one (see e.g.* [10]*).*

If M is the Q–representation shown in line 5 we consider $Gr_{(1)}(M)$ which consists of the lines of k^4 invariant under the linear operators $E_{21}+E_{43}$ and E_{32}. It is easy to see that this set consists only of the line spanned by the fourth basis vector. Similarly if M is the Q–representation shown in the last row of table 1, $Gr_{(1)}(M)$ consists only of one point: the line spanned by the third basis vector.

We notice that the quiver Grassmannian $Gr_{\mathbf{e}}(M)$ is closed inside the product $\prod_{i\in Q_0} Gr_{e_i}(M(i))$, where $Gr_{e_i}(M(i))$ denotes the usual Grassmannian of all vector subspaces of $M(i)$ of dimension e_i, which is a projective variety. As a consequence, $Gr_{\mathbf{e}}(M)$ is a complex projective variety. We denote by $\chi_{\mathbf{e}}(M)$ its Euler–Poincaré characteristic. In the examples shown above $\chi_{\mathbf{e}}(M)$ is one if $Gr_{\mathbf{e}}(M)$ is a (double) point and zero if it is the empty set.

1.2. **The main result.** The following theorem is our main result.

Theorem 1. *Let M be a Q–representation and for every $i \in Q_0$ let $B(i)$ be a linear basis of $M(i)$ such that for every arrow $a : j \to i$ of Q and every element $b \in B(j)$ there exists an element $b' \in B(i)$ and $c \in k$ (possibly zero) such that*

$$M(a)b = cb'. \tag{1}$$

Suppose that each $v \in B(i)$ and all its multiples cv, $c \in k^*$, is assigned a degree $d(cv) = d(v) \in \mathbb{Z}$ so that:

(D1) *for all $i \in Q_0$ all vectors from $B(i)$ have different degrees;*
(D2) *for every arrow $a : j \to i$ of Q, whenever $b_1 \neq b_2$ are elements of $B(j)$ such that $M(a)b_1$ and $M(a)b_2$ are non-zero we have:*

$$d(M(a)b_1) - d(M(a)b_2) = d(b_1) - d(b_2). \tag{2}$$

Then

$$\chi_{\mathbf{e}}(M) = |\{N \in Gr_{\mathbf{e}}(M) : N(i) \text{ is spanned by a part of } B(i)\}|. \tag{3}$$

The hypothesis (1) says that every column and every row of the matrix $M(a)$ contains at most one entry different from zero.

The hypothesis $(D2)$ can be replaced by saying that every arrow a of Q has a degree $d(a) \in \mathbb{Z}$ so that $d(b') = d(b) + d(a)$ whenever $M(a)b = cb'$, for some non–zero coefficient $c \in k$.

The thesis (3) says that we need to count the number of “coordinate” subrepresentations i.e. those $N \in Gr_{\mathbf{e}}(M)$ whose vector space $N(i)$ is a coordinate subspace in the basis $B(i)$ (i.e. is spanned by elements of $B(i)$).

Example 3. *Let Q be the quiver with only one vertex and no–arrows. A Q–representation is just a vector space V and the quiver Grassmannians are*

usual Grassmannians of vector subspaces. Let $\{v_1, \cdots, v_n\}$ be a basis of V. We assign degree $d(v_i) := i$ and the hypotheses of theorem 1 are satisfied. Then, by theorem 1, $\chi(Gr_k(V))$ is the number of coordinate vector subspaces (i.e. generated by basis vectors) of V of dimension k. We hence find the well–known result: $\chi(Gr_k(V)) = \binom{n}{k}$.

Let us give other examples with the help of table 1. The Q–representations shown in line 1 are isomorphic, but the first one does not satisfy the hypothesis (1) *and we cannot apply theorem 1, while the second one does.*

The second line shows an interesting example. The Q–representation M of this line is a "deformation" of $M' := k \overset{1}{\underset{0}{\leftleftarrows}} k$ and they have the same quiver Grassmannians (see lemma 1.4). These two Q–representations are indeed right–equivalent *in the sense of* [13]*. Theorem 1 applies to M' and we can hence compute $\chi_{\mathbf{e}}(M)$.*

In line 3 of table 1 we choose $d(a) = d(b) := 0$ and $d(c) := 1$ and hence the choice of a degree for the generator of the one–dimensional vector space at vertex 3 *determines the choice of a degree for the two basis vectors at vertices* 2 *and* 3 *and these two degrees are different. We can hence apply theorem 1.*

In line 4 we choose $d(a) := 0$ and $d(b) := 1$.

In line 5 we choose $d(a) = d(b) = 1$.

In line 6 we choose $d(a) = 1$ and $d(b) = 2$.

1.3. **Coefficient–quiver.** In order to compute $\chi_{\mathbf{e}}(M)$ with the help of theorem 1 one can use a combinatorial tool called the *coeffcient–quiver* $\tilde{Q}(M, B)$ of M in the basis B (introduced by Ringel in [24]). Let us recall its definition and show its utility. Let M be a Q–representation and $B = \cup_{i \in Q_0} B(i)$ a collection of basis $B(i)$ of $M(i)$. The set B is hence a basis of the vector space $\bigoplus_{i \in Q_0} M(i)$ and we refer to it as a *basis* of M. The coefficient–quiver $\tilde{Q}(M, B)$ is a quiver whose vertices are identified with the elements of B; the arrows are defined as follows: for every arrow $a : j \to i$ of Q and every element $b \in B(j)$ we expand $M(a)b = \sum c_{b'} b'$ in the basis $B(i)$ of $M(i)$ and we put an arrow (still denoted by a) from b to $b' \in B(i)$ in $\tilde{Q}(M, B)$ if the coefficient $c_{b'}$ of b' in this expansion is non–zero. Table 1 shows examples of coefficient–quivers (which are denoted simply by $\tilde{Q}(M)$ since they are in the basis in which M is presented).

We denote by $T \overrightarrow{\subset} \tilde{Q}(M)$ a *successor closed* subquiver T of $\tilde{Q}(M)$, i.e. a subquiver T such that if $j \in T_0$ is one of its vertices and $a : j \to i$ is an arrow of Q then a is an arrow of T.

It is easy to see that the following proposition is equivalent to theorem 1.

Proposition 1. *Let M be a Q–representation satisfying hypotheses of theorem 1. Then*

$$\chi_{\mathbf{e}}(M) = |\{T \overrightarrow{\subset} \tilde{Q}(M) \ : \ |T_0 \cap B(i)| = e_i, \, \forall i \in Q_0\}| \tag{4}$$

where T_0 denotes the vertices of T. In particular $\chi_{\mathbf{e}}(M)$ is positive.

For example let us consider the Q–representation M shown in the third line of table 1. We have already noticed that M satisfies hypotheses of theorem 1. Then we apply proposition 1 and we find $\chi_{(1,0,0)}(M) = 2$. Indeed

there are two successor closed subquivers of $\tilde{Q}(M)$ with $|T_0 \cap B(1)| = 2$ and $|T_0 \cap B(2)| = |T_0 \cap B(3)| = 0$ which are the two sinks (this is consistent with the fact that $Gr_{(1,0,0)}(M) = \mathbb{P}^1(k^2)$ is a projective line). Many other examples can be taken from table 1.

1.4. **String–modules.** We now show a class of Q–representations which satisfy the hypotheses of theorem 1.

A Q–representation M is called a *string module* if it admits a basis B_0 such that the coefficient–quiver $\tilde{Q}(M, B_0)$ in this basis is a chain (i.e. a 2–regular graph not necessarily connected) and if every column and every row of every matrix $M(a)$ in this basis B_0 has at most one non–zero entry, i.e. it satisfies (1). We remark that this definition follows [12] but not [24] where (1) is not required. For a string module M we sometimes avoid mentioning the basis B_0 and we denote the corresponding coefficient–quiver simply by $\tilde{Q}(M)$. The Q–representations shown in table 1 are all string modules except the second one. It can be shown that a string module M is indecomposable if and only if $\tilde{Q}(M)$ is connected ([12], [19, Sec. 3.5 and 4.1]).

Given an indecomposable string module M, the chain $\tilde{Q}(M)$ has two extreme vertices (i.e. joined with exactly one vertex). We say that two arrows of $\tilde{Q}(M)$ have the *same orientation* if they both point toward the same extreme vertex and they have *different orientation* otherwise. For example the two arrows labelled by a in lines 5 and 6 of table 1 have the same orientation in the line 5 while they have different orientation in the line 6.

During private conversations with J. Schröer we were introduced to the following definition.

Definition 1.1. *A string module M is called* orientable *if for every arrow a of Q, all the corresponding arrows a of $\tilde{Q}(M)$ have the same orientation.*

For example line 5 of table 1 shows an orientable string module while the line 6 shows a non–orientable one.

Proposition 2. *If M is an orientable string module then* (4) *holds.*

In section 2 we show that an orientable string module satisfies (1), $(D1)$ and $(D2)$ and hence, by proposition 1, they satisfy (4).

1.5. **Explicit computations in type $\tilde{A}_{p,1}$.** In this section we compute explicitly $\chi_{\mathbf{e}}(M)$ for some indecomposable representation M of the affine quiver $Q_{p,1}$ of type $\tilde{A}_{p,1}$. Let us recall the definition of $Q_{p,1}$.

Let $p \geq 1$ be an integer. By definition $Q_{p,1}$ has one sink, one source and $p+1$ arrows which form two paths, one with p arrows and the other with one arrow. We denote the vertices of $Q_{p,1}$ by numbers from 1 to $p+1$ so that 1 is the sink, $p+1$ is the source and k is joined to $k+1$ by the arrow ε_k, for $k = 1, 2, \cdots, p$ and $p+1$ is joined to 1 by the arrow ε_0 as shown below:

$$
\begin{tikzcd}
 & 2 \arrow[d, "\varepsilon_1"'] & \cdots \arrow[l, "\varepsilon_2"'] & p \arrow[l, "\varepsilon_{p-1}"'] \\
Q_{p,1} := & 1 & & p+1 \arrow[ll, "\varepsilon_0"] \arrow[u, "\varepsilon_p"']
\end{tikzcd}
$$

For every $n \geq 0$ and $1 \leq t \leq p$ we define the $Q_{p,1}$–representations

$$
M_p^n(\overline{[1,t]}) := \begin{tikzcd}
\boxed{k^{n+1}} \arrow[d] & \\
k^{n+1} \arrow[d] & k^n \arrow[ul, "\varphi_1"'] \\
\vdots \arrow[d] & \vdots \arrow[u] \\
k^{n+1} & k^n \arrow[l, "\varphi_2"] \arrow[u]
\end{tikzcd}
\qquad
M_p^n(\underline{[1,t]}) := \begin{tikzcd}
\boxed{k^n} \arrow[d] & \\
k^n \arrow[d] & k^{n+1} \arrow[ul, "\varphi_2^t"'] \\
\vdots \arrow[d] & \vdots \arrow[u] \\
k^n & k^{n+1} \arrow[l, "\varphi_1^t"] \arrow[u]
\end{tikzcd}
$$

where the highlighted vector spaces correspond to the vertex t. These representations are called respectively pre–projective and pre–injective modules (see e.g.[2]).

For every $\lambda \in k$ and $n \geq 1$, let $Reg_p^n(\lambda)$ be the $Q_{p,1}$–representation

$$
Reg_p^n(\lambda) := \begin{tikzcd}
k^n \arrow[d, "="'] & \cdots \arrow[l, "="'] & k^n \arrow[l, "="'] \\
k^n & & k^n \arrow[ll, "J_n(\lambda)"] \arrow[u, "="']
\end{tikzcd}
$$

with a Jordan block $J_n(\lambda)$ of eigenvalue λ at the arrow ε_0 and the identity map in all the other arrows. This reprsentation is called regular homogeneous. It is easy to see that $M_p^n(\overline{[1,t]})$, $M_p^n(\underline{[1,t]})$ and $Reg_p^n(0)$ are orientable string modules (see lemma 1.2) and $\chi_{\mathbf{e}}(Reg_p^n(\lambda)) = \chi_{\mathbf{e}}(Reg_p^n(0))$ for every $\lambda \in k$ (section 4.2). We can hence apply theorem 1 (or proposition 2).

We often use the following notation:

$$
\chi_{\mathbf{e}}([r,s]) := \prod_{k=r}^{s-2} \binom{e_k - e_s}{e_{k+1} - e_s} = \prod_{k=r+1}^{s-1} \binom{e_r - e_{k+1}}{e_k - e_{k+1}} \tag{5}
$$

with the convention that this product equals one whenever $r > s-2$. We interpret $\chi_{\mathbf{e}}([r,s])$ as the Euler characteristic of the flag variety

$$\{k^{e_r} \supseteq M_{r+1} \supseteq \cdots \supseteq M_{s-1} \supseteq k^{e_s} \mid dim(M_k) = e_k\}.$$

Proposition 3. *For every $n \geq 1$, $1 \leq t \leq p$ and $\lambda \in k$ we have*

$$
\chi_{(e_1,\cdots,e_{p+1})}(M_p^n(\overline{[1,t]})) = \tag{6}
$$

$$
\binom{e_1 - 1}{e_{p+1}} \binom{n+1-e_t}{e_1 - e_t} \binom{n+1-e_{t+1}}{e_t - e_{t+1}} \binom{n - e_{p+1}}{e_{t+1} - e_{p+1}} \chi_{\mathbf{e}}([1,t]) \chi_{\mathbf{e}}([t+1,p+1])
$$

$$
\chi_{(e_1,\cdots,e_{p+1})}(M_p^n(\underline{[1,t]})) = \tag{7}
$$

$$
\binom{n - e_{p+1}}{e_1 - e_{p+1}} \binom{e_{t+1}}{e_{p+1}} \binom{e_t + 1}{e_{t+1}} \binom{e_1}{e_t} \chi_{\mathbf{e}}([1,t]) \chi_{\mathbf{e}}([t+1,p+1])
$$

$$
\chi_{\mathbf{e}}(Reg_p^n(\lambda)) = \binom{e_1}{e_{p+1}} \binom{n - e_{p+1}}{e_1 - e_{p+1}} \chi_{\mathbf{e}}([1,p+1]) \tag{8}
$$

We always use the convention that the binomial coefficient $\binom{p}{q}$ equals 0 if $q < 0$, $p < 0$, $q > p$ and it equals 1 if $q = 0$ and $p \geq q$.

2. Proof of theorem 1

The proof is based on the following well–known fact: given a complex projective variety X and an algebraic action $\varphi : T \times X \to X$, $(\lambda, x) \mapsto \lambda.x$ of the one–dimensional torus $T = \mathbb{C}^*$ with finitely many fixed points, then the number of fixed points equals the Euler–Poincaré characteristic $\chi(X)$ of X. To see this we consider the decomposition $X = X^T \coprod Y$ of X into the disjoint union of the set X^T of fixed points of φ and of their complement $Y := X \setminus X^T$. Such sets are locally closed and hence $\chi(X) = \chi(X^T) + \chi(Y)$. The restriction of φ to Y defines a surjective morphism $\varphi : T \times Y \to Y$ whose fibers are all isomorphic to $\mathbb{C}^*$. It follows that $\chi(Y) = \chi(\mathbb{C}^*) = 0$ and hence $\chi(X) = \chi(X^T)$ which equals the number of fixed points of φ.

We hence find a torus action on our quiver Grassmannians.

Let M be a representation satisfying hypotheses (D1) and (D2) of the theorem. The torus k^* acts on M as follows:

$$\lambda.b := \lambda^{d(b)} b, \qquad \lambda \in k^* \tag{9}$$

for every element $b \in B$ of the basis B extended by linearity to all the elements of M. This action extends to quiver Grassmannians:

Lemma 1.1. *Let $U \in Gr_{\mathbf{e}}(M)$ be a subrepresention of M of dimension vector $\mathbf{e}$. Then, given $\lambda \in \mathrm{k}^*$, the set $\lambda.U := \{\lambda.u | \, u \in U\}$ is a subrepresentation of M of the same dimension vector $\mathbf{e}$ of U.*

Proof. Given an arrow $a : j \to i$ of Q we define the number $d(a) := d(M(a)b) - d(b)$ for an element $b \in B(j)$ such that $M(a)b$ is non–zero. This definition is independent of the choice of b in view of (D2). Then it is easy to verify that for every $v \in M(j)$

$$\lambda.(M(a)v) = \lambda^{d(a)} M(a)(\lambda.v)$$

which concludes the proof. □

Given a subrepresentation $U \in Gr_{\mathbf{e}}(M)$, the element $\lambda \in k^*$ acts on each vector subspace $U(i)$ as a diagonal operator with different eigenvalues, in view of property (D1). Then the fixed subrepresentations $U = \lambda.U \in Gr_{\mathbf{e}}(M)$ are precisely the coordinate subspaces of M in the basis B of dimension $e := \sum_i e_i$ which concludes the proof of theorem 1.

3. Proof of proposition 2

We prove that an orientable string module M satisfies the hypotheses of theorem 1. By definition there exists a basis B_0 of M so that (1) is satisfied and the coefficient–quiver $\tilde{Q}(M, B_0)$ in B_0 is a chain. We have to assign a degree $d(b) \in \mathbb{Z}$ to the elements of B_0 (which are also the vertices of $\tilde{Q}(M, B_0)$) so that (D1) and (D2) are satisfied.

Since $S := \tilde{Q}(M, B_0)$ is a chain we number the vertices of S as $s_1, s_2, \cdots$ in such a way that for every $i = 1, \cdots, m$ there is a unique edge ε_i between s_i and s_{i+1}. We assign the degree $d(s_i) := i$ for $i = 1, 2, \cdots$. Then (D1) is clearly satisfied (all the elements of B_0 have different degrees and hence all the elements of $B_0(i)$ have different degrees). Since M is orientable it is also easy to prove that (D2) is satisfied. Indeed, by definition, for every arrow

a of Q all the corresponding arrows a of S have all the same orientation, either all of them are oriented from s_i to s_{i+1} or from s_{i+1} to s_i.

4. Proof of proposition 3

For the convenience of the reader we prove proposition 3 first in the case $p = 1$ (the Kronecker quiver) and hence for $p \geq 1$.

All the proofs are based on the following lemma.

Lemma 1.2. $M_p^n(\overline{[1,t]})$, $M_p^n(\underline{[1,t]})$ *and* $Reg_p^n(0)$ *are orientable string modules (in the sense of definition 1.1). In particular* (4) *holds.*

Proof. All the linear maps defining such $Q_{p,1}$–representations satisfy (1). It remains to show that their coefficient–quiver is a chain.

Let S_{ε_0} be the subquiver of $Q_{p,1}$ obtained by removing the arrow ε_0. We join together n copies of S_{ε_0} by using the arrow ε_0 and we get a string that we denote by S_0^n. The coefficient–quiver of $Reg_p^n(0)$ is S_0^n which is a chain.

Let $1 \leq t \leq p$ be a vertex of $Q_{p,1}$. We consider the full subquiver $S(\overline{[1,t]})$ of $Q_{p,1}$ with vertex set all the vertices $1, 2, \cdots, t$. We join the string S_0^n with the string $S([1,t])$ by using the arrow ε_0 and we get a new string that we call $S^n(\overline{[1,t]})$. Such a string is the coefficient–quiver of $M_p^n(\overline{[1,t]})$

In order to get the coefficient–quiver of $M_p^n(\underline{[1,t]})$ we proceed similarly: we consider the full subquiver $S(\underline{[1,t]})$ with vertices $t+1, t+2, \cdots, p, p+1$. We join $S(\underline{[1,t]})$ with S^n by using the arrow ε_0 and we get a quiver $S^n(\underline{[1,t]})$. Such a quiver is the coefficient–quiver of $M_p^n(\underline{[1,t]})$. Figure 1 shows the case $p = 4$, $t = n = 3$.

□

4.1. **Type $\tilde{A}_{1,1}$: the Kronecker quiver.** In this section we consider the Kronecker quiver $Q_{1,1} := 1 \overset{\varepsilon_1}{\underset{\varepsilon_0}{\leftleftarrows}} 2$ and its representations over the field k of complex numbers. Let $\varphi_1, \varphi_2 : k^n \to k^{n+1}$ be respectively the immersion in the vector subspace spanned respectively by the first and by the last n basis vectors. For every $n \geq 0$ and $\lambda \in k$ we consider the representations

$$M_1^n(\overline{[1,1]}) := k^{n+1} \overset{\varphi_1}{\underset{\varphi_2}{\leftleftarrows}} k^n; \quad M_1^n\underline{[1,1]} := k^n \overset{\varphi_1^t}{\underset{\varphi_2^t}{\leftleftarrows}} k^{n+1}$$

$$Reg_1^n(\lambda) := \; k^n \overset{=}{\underset{J_n(\lambda)}{\leftleftarrows}} k^n.$$

The next result is contained in [9]. We give a slightly different proof by using theorem 1.

Proposition 4. [9, Propositions 4.3 and 5.3] *For every dimension vector* $\mathbf{e} = (e_1, e_2)$ *and* $n \geq 0$ *we have:*

$$\chi_{(e_1,e_2)}(M_1^n(\overline{[1,1]})) = \binom{n+1-e_2}{n+1-e_1}\binom{e_1-1}{e_2} + \delta_{e_1,0}\delta_{e_2,0} \tag{10}$$

$$\chi_{(e_1,e_2)}(M_1^n(\underline{[1,1]})) = \binom{e_1+1}{e_2}\binom{n-e_2}{n-e_1} + \delta_{e_1,n}\delta_{e_2,n+1} \tag{11}$$

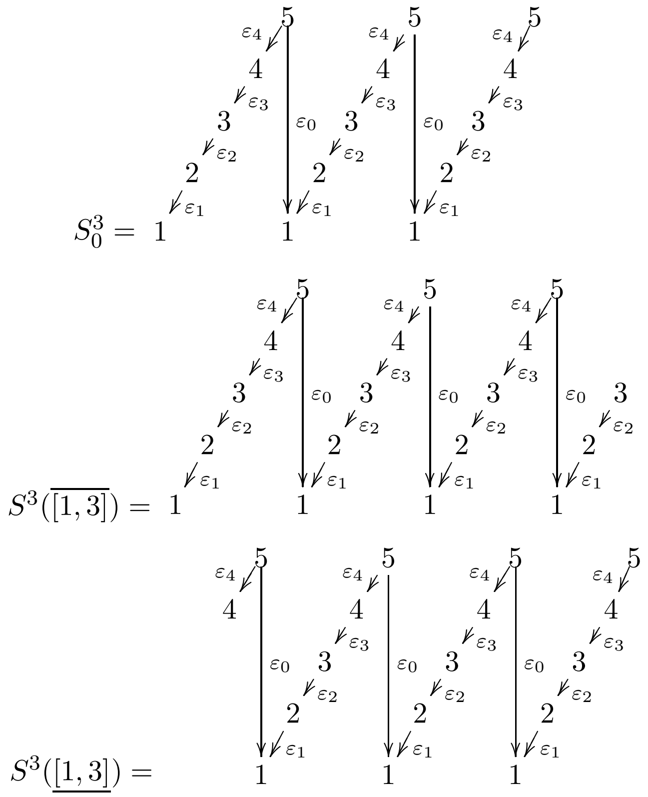


FIGURE 1. The coefficient–quiver of $Reg_4^3(0)$, $M_4^3(\overline{[1,3]})$ and $M_4^3(\underline{[1,3]})$ respectively

where $\delta_{a,b}$ denotes the Kronecker delta. For every $\lambda \in k$:

$$\chi_{(e_1,e_2)}(Reg_1^n(\lambda)) = \binom{n-e_2}{n-e_1}\binom{e_1}{e_2} \tag{12}$$

Proof. We notice that (11) follows from (10). Indeed $M_1^n\underline{[1,1]} \simeq DM_1^n(\overline{[1,1]})$ where $D = Hom_k(\cdot, k)$ is the duality functor and the isomorphism follows by exchanging the two vertices. Then we have (see also [8, Sec. 1.2]):

$$\chi_{(e_1,e_2)}(M_1^n(\underline{[1,1]})) = \chi_{(n+1-e_2,n-e_1)}(M_1^n(\overline{[1,1]})).$$

We hence prove (10). By lemma 1.2, the representation $M_1^n(\overline{[1,1]})$ is an orientable string module and we can apply theorem 1. In order to compute $\chi_{(e_1,e_2)}(M_1^n(\overline{[1,1]}))$, we have hence to count couples $\{T_1, T_2\}$ of subsets $T_1 \subset [1, n+1]$, $T_2 \subset [1, n]$ such that $|T_i| = e_i$ $(i = 1, 2)$ and $\varphi_1(T_2) \subset T_1$, $\varphi_2(T_2) \subset T_1$ where $\varphi_1, \varphi_2 : [1, n] \to [1, n+1]$ are the two maps defined by $\varphi_1(k) = k$ and $\varphi_2(k) = k+1$ for $k = 1, 2, \cdots, n$ (here and in the sequel we use the notation $[1, m] := \{1, 2, \cdots, m\}$). We need the following lemma.

Lemma 1.3. [9, proof of proposition 4.3] *Let n and r be positive integers such that $1 \leq r \leq n$. For an r–element subset J of $[1, n]$ we denote by*

$c(J)$ the number of connected components of J (i.e. the number of maximal connected intervals in J). The number of r–element subsets J of $[1,n]$ such that $c(J)=c$ is $\binom{r-1}{c-1}\binom{n+1-r}{c}$.

Proof. A proof of lemma 1.3 can be found in [9, proof of proposition 4.3] □

We hence continue the proof of (10). The choice of an element $k\in[1,n]$ determines the choice of the two different elements $\varphi_1(k)$ and $\varphi_2(k)$ of $[1,n+1]$; in general the choice of a subset T_2 of $[1,n]$ of cardinality e_2 with c connected components determines the choice of $c+e_2$ elements of $[1,n+1]$. Given such a set T_2, there are hence $\binom{n+1-(c+e_2)}{e_1-(c+e_2)}$ choices for the sets T_1 such that $\{T_1,T_2\}$ is a desired couple. If $e_1=e_2=0$ then $\chi_{(0,0)}(M_1^n(\overline{[1,1]}))=1$. We assume $e_1\geq e_2\geq 1$. By lemma 1.3 the number of e_2–element subsets T_2 of $[1,n]$ with $c(T_2)=c$ equals $\binom{e_2-1}{c-1}\binom{n+1-e_2}{c}$. The number of desired couples $\{T_1,T_2\}$ is hence

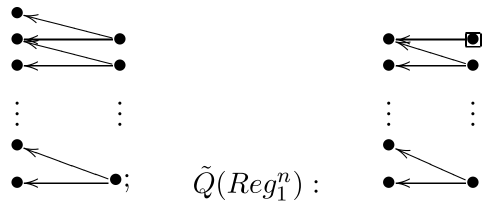
$$\begin{aligned}\chi_{(e_1,e_2)}(M_1^n(\overline{[1,1]})) &= \sum_{c=1}^{e_1-e_2}\binom{n+1-(c+e_2)}{e_1-(c+e_2)}\binom{e_2-1}{c-1}\binom{n+1-e_2}{c}\\ &= \sum_{c=1}^{e_1-e_2}\binom{e_1-e_2}{c}\binom{e_2-1}{c-1}\binom{n+1-e_2}{e_1-e_2}\\ &= \binom{n+1-e_2}{e_1-e_2}\sum_{c=1}^{e_1-e_2}\binom{e_1-e_2}{c}\binom{e_2-1}{e_2-c}\\ &= \binom{n+1-e_2}{e_1-e_2}\binom{e_1-1}{e_2}\end{aligned}$$

In the second equality we have used the identity: $\binom{n+1-r-q}{p-q}\binom{n+1-r}{q}=\binom{p}{q}\binom{n+1-r}{p}$ with $q=c$, $p=e_1-e_2$ and $r=e_2$; in the last equality we have used the Vandermonde's identity: $\sum_k\binom{a}{k}\binom{b}{c-k}=\binom{a+b}{c}$.

We now prove (12). We first assume that $\lambda=0$. The representation $Reg_{1,1}^n(0)$ is an orientable string module and we apply theorem 1. We prove (12) by induction on $n\geq 1$. For $n=0$ it is clear. Let hence $n\geq 1$. We have hence to count the number of couples $\{T_1,T_2\}$ of subsets $T_2\subset T_1\subset[1,n]$ such that $|T_i|=e_i$ and $J_n(0)T_2\subset T_1\cup 0$ where $J_n(0):[1,n]\to[1,n]\cup\{0\}$ maps k to $k-1$ for $k=1,2,\cdots,n$. Alternatively, by proposition 1, we can consider the coefficient quiver $\tilde{Q}(Reg_1^n)$ of Reg_1^n (shown in figure 2) and count its successor closed subquivers with e_1 sources and e_2 sinks. Such a subquiver either contains the unique vertex of $\tilde{Q}(Reg_1^n)$ which is the source of a unique arrow (highlighted in figure 2) or it does not. Alternatively either T_2 contains $1=Ker(J_n(0))$ or it does not. We hence have

$$\begin{aligned}\chi_{(e_1,e_2)}(Reg_1^n(0)) &= \chi_{(e_1-1,e_2-1)}(Reg_1^{n-1}(0))+\chi_{(e_1,e_2)}(M_1^{n-1}(\overline{[1,1]}))\\ &= \binom{n-e_2}{n-e_1}\binom{e_1-1}{e_2-1}+\binom{n-e_2}{n-e_1}\binom{e_1-1}{e_2}+\delta_{e_1,0}\delta_{e_2,0}\\ &= \binom{n-e_2}{n-e_1}\binom{e_1}{e_2}.\end{aligned}$$

and we are done (we use the obvious fact that $\binom{a-1}{b-1}+\binom{a-1}{b}=\binom{a}{b}-\delta_{a,0}\delta_{b,0}$).



$\tilde{Q}(M_1^n(\overline{[1,1]}))$: ; $\tilde{Q}(Reg_1^n)$:

$M_1^n(\overline{[1,1]})$: $k^{n+1} \underset{\varphi_2}{\overset{\varphi_1}{\leftleftarrows}} k^n$; Reg_1^n : $k^n \underset{=}{\overset{J_n(0)}{\leftleftarrows}} k^n$

FIGURE 2. Coefficient–quiver of $Q_{1,1}$–representations.

It remains to be considered the case where $\lambda \neq 0$ which is solved in the following lemma.

Lemma 1.4. *For every $\lambda \in \mathbb{C}$ and $n \geq 1$ we have*

$$\chi_{\mathbf{e}}(Reg_1^n(\lambda)) = \chi_{\mathbf{e}}(Reg_1^n(0))$$

Proof. As vector spaces, $Reg_1^n(0)$ and $Reg_1^n(\lambda)$ are isomorphic to k^{2n}. The path algebra $kQ_{1,1}$ acts on these isomorphic vector spaces by two actions that we denote respectively by $*$ and $\circ$. We consider the automorphism ψ of the path algebra $kQ_{1,1}$ which sends ε_0 to $\varepsilon_0 + \lambda\varepsilon_1$. For every σ in $kQ_{1,1}$ and every m in $Reg_{1,1}^n(0)$, $\psi(\sigma) * m = \sigma \circ m$. Roughly speaking what the automorphism ψ does is the following: the arrow ε_0 acts as $J_n(0)$ on $Reg_{1,1}^n(0)$, while the arrow ε_1 acts as the identity. Then $\psi(\varepsilon_0)$ acts as $J_n(0) + \lambda Id = J_n(\lambda)$. With this action $Reg_{1,1}^n(0)$ is isomorphic to $Reg_{1,1}^n(\lambda)$ (as kQ_{11}–module). In particular the two representations have the same quiver Grassmannians. This proves that they are right–equivalent in the sense of [13]. $\square$

This concludes the proof of proposition 4. $\square$

4.2. **Type $\tilde{A}_{p,1}$.** We prove proposition 3 for every $p \geq 2$. The duality functor D sends a representation of $Q_{p,1}$ to a representation of the opposite quiver $Q_{p,1}^{op}$. The symmetries of such quiver induce an isomorphism $M_p^n(\underline{[1,t]}) \simeq DM_p^n(\overline{[1,p+1-t]})$ and, for every dimension vector $\mathbf{e} = (e_1, \cdots, e_{p+1})$, we have:

$$\chi_{\mathbf{e}}(M_p^n(\underline{[1,t]})) = \chi_{(d_{p+1}-e_{p+1},\cdots,d_1-e_1)}(M_p^n(\overline{[1,p+1-t]}))$$

where $\mathbf{d} = (d_1, \cdots, d_{p+1})$ is the dimension vector of $M_p^n(\underline{[1,t]})$. Then (7) follows from (6).

We prove (6). By lemma 1.2, the representation $M_p^n(\overline{[1,t]})$ satisfies the hypotheses of theorem 1. In order to compute $\chi_{\mathbf{e}}(M_p^n(\overline{[1,t]}))$ we hence have to count sets $\{T_1, \cdots, T_{p+1}\}$ of subsets $T_1, \cdots, T_t \subset [1, n+1]$, $T_{t+1}, \cdots, T_{p+1} \subset [1, n]$ such that: $|T_i| = e_i$ and $\varphi_1(T_{t+1}) \subset T_t$, $\varphi_2(T_{p+1}) \subset T_1$ and $T_k \subset T_{k-1}$ ($k \neq t+1$, $k \neq p+1$) where $\varphi_1, \varphi_2 : [1, n] \to [1, n+1]$ are defined by $\varphi_1(k) := k$ and $\varphi_2(k) := k+1$ for every $k = 1, \cdots n$.

For a choice of the quadruple $\{T_1, T_t, T_{t+1}, T_{p+1}\}$ (this set could collapse to a quadruple in which two elements coincide but it does not make any difference in the sequel and we still refer to it as a quadruple) there are $\chi_{\mathbf{e}}([1,t])$ choices for $\{T_2, \cdots, T_{t-1}\}$ and $\chi_{\mathbf{e}}([t+1, p+1])$ choices for $\{T_{t+2}, \cdots, T_p\}$ such that $\{T_1, \cdots, T_{p+1}\}$ is a desired tuple.

We hence prove that the number of quadruples $\{T_1, T_t, T_{t+1}, T_{p+1}\}$ equals:

$$\binom{e_1 - 1}{e_{p+1}}\binom{n+1-e_t}{e_1 - e_t}\binom{n+1-e_{t+1}}{e_t - e_{t+1}}\binom{n-e_{p+1}}{e_{t+1} - e_{p+1}} \tag{13}$$

from which (6) follows. We hence have to count the number of quadruples $\{T_1, T_t, T_{t+1}, T_{p+1}\}$ of subsets $T_t \subset T_1 \subset [1, n+1]$, $T_{p+1} \subset T_{t+1} \subset [1, n]$ such that $|T_i| = e_i$, $\varphi_1(T_{t+1}) \subset T_t$ and $\varphi_2(T_{p+1}) \subset T_1$.

We need the following lemma.

Lemma 1.5. *Let n and e be positive integers such that $1 \leq e \leq n$. As before, we denote by $c(J)$ the number of connected components of an e–element subset J of $[1,n]$. For every integer c, we have*

(1) *the number of e–element subsets J of $[1,n]$ such that $c(J) = c$ and J contains n, is $\binom{e-1}{c-1}\binom{n-e}{c-1}$;*
(2) *the number of e–element subsets J of $[1,n]$ such that $c(J) = c$ and J does not contain n is $\binom{e-1}{c-1}\binom{n-e}{c}$;*
(3) *for every $0 \leq r \leq q \leq p$, $\binom{p}{q}\binom{q}{r} = \binom{p}{r}\binom{p-r}{q-r}$.*

Proof. The proof of lemma 1.5 follows from lemma 1.3 by an easy induction. □

$$M_p^n(\overline{[1,t]}) := \begin{tikzcd} k^{n+1} \arrow[d] & \\ k^{n+1} \arrow[d] & k^n \arrow[ul, "\varphi_1"'] \\ \vdots \arrow[d] & \vdots \arrow[u] \\ k^{n+1} & k^n \arrow[l, "\varphi_2"] \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} {[1,n+1]} \supset T_t \arrow[ddd] & \\ & T_{t+1} \subset [1,n] \arrow[ul, hook, "\varphi_1"'] \\ & \\ {[1,n+1]} \supset T_1 & T_{p+1} \subset [1,n] \arrow[l, hook, "\varphi_2"] \arrow[uu, hook] \end{tikzcd}$$

Let T_{p+1} be an e_{p+1}–element subset of $[1,n]$ and let us count the number of desired quadruples $\{T_1, T_t, T_{t+1}, T_{p+1}\}$ containing T_{p+1}. We notice that T_1 contains both $\varphi_1(T_{p+1})$ and $\varphi_2(T_{p+1})$. In particular, if c denotes the number of connected components of T_{p+1}, then T_1 must contain $c + e_{p+1}$ elements of $[1, n+1]$. We distinguish the two cases: either T_{p+1} contains n or it does not.

(1) If T_{p+1} contains n (by lemma 1.5 there are $\binom{e_{p+1}-1}{c-1}\binom{n-e_{p+1}}{c-1}$ choices for such subsets) then every possible T_1 contains the element $\varphi_2(n) = (n+1)$. Then there are $\binom{n+1-c-e_{p+1}}{e_1-c-e_{p+1}}$ choices for T_1. Now either T_t contains $(n+1)$ or it does not. If it contains $(n+1)$ (there are $\binom{e_1-1-e_{p+1}}{e_t-1-e_{p+1}}$ choices for such sets) then there are $\binom{e_t-1-e_{p+1}}{e_{t+1}-e_{p+1}}$ choices for T_{t+1}; if T_t does not contain $(n+1)$ (there are $\binom{e_1-1-e_{p+1}}{e_t-e_{p+1}}$ choices for such sets) then there are $\binom{e_t-e_{p+1}}{e_{t+1}-e_{p+1}}$ choices for T_{t+1}.

The number of quadruples $\{T_1, T_t, T_{t+1}, T_{p+1}\}$ such that T_{p+1} contains n is hence given by:

$$\begin{aligned}
&\sum_c \binom{e_{p+1}-1}{c-1}\binom{n-e_{p+1}}{c-1}\binom{n-e_{p+1}-(c-1)}{e_1-e_{p+1}-c}\cdot \\
&[\binom{e_1-1-e_{p+1}}{e_t-1-e_{p+1}}\binom{e_t-1-e_{p+1}}{e_{t+1}-e_{p+1}}+\binom{e_1-1-e_{p+1}}{e_t-e_{p+1}}\binom{e_t-e_{p+1}}{e_{t+1}-e_{p+1}}] = \\
&\sum_c \binom{e_{p+1}-1}{c-1}\binom{n-e_{p+1}}{c-1}\binom{n-e_{p+1}-(c-1)}{e_1-e_{p+1}-c}\cdot \\
&[\binom{e_1-1-e_{p+1}}{e_{t+1}-e_{p+1}}\binom{e_1-1-e_{t+1}}{e_t-1-e_{t+1}}+\binom{e_1-1-e_{p+1}}{e_{t+1}-e_{p+1}}\binom{e_1-1-e_{t+1}}{e_t-e_{t+1}}] = \\
&\sum_c \binom{e_{p+1}-1}{c-1}\binom{e_1-e_{p+1}-1}{c-1}\binom{n-e_{p+1}}{e_1-e_{p+1}-1}\binom{e_1-1-e_{p+1}}{e_{t+1}-e_{p+1}}\binom{e_1-e_{t+1}}{e_t-e_{t+1}} =
\end{aligned}$$

$$\binom{e_1-2}{e_1-e_{p+1}-1}\binom{n-e_{p+1}}{e_1-e_{p+1}-1}\binom{e_1-e_{t+1}}{e_t-e_{t+1}}\binom{e_1-1-e_{p+1}}{e_{t+1}-e_{p+1}} \tag{14}$$

In the first and third equality we have used part (3) of lemma 1.5; in the last equality we have used the Vandermonde's identity.

(2) If T_{p+1} does not contain n (by lemma 1.5 there are $\binom{e_{p+1}-1}{c-1}\binom{n-e_{p+1}}{c}$ choices for such sets) then either T_1 contains $(n+1)$ or it does not. Since T_{p+1} does not contain n, there are $\binom{n-c-e_{p+1}}{e_1-1-c-e_{p+1}}$ choices of sets T_1 containing $(n+1)$. In this case either T_t contains $(n+1)$ (there are $\binom{e_1-1-e_{p+1}}{e_t-1-e_{p+1}}$ choices of such sets) or it does not (there are $\binom{e_1-1-e_{p+1}}{e_t-e_{p+1}}$ choices of such sets). If T_t contains $(n+1)$ then there are $\binom{e_t-1-e_{p+1}}{e_{t+1}-e_{p+1}}$ choices for T_{t+1}. If T_t does not contain $(n+1)$ then there are $\binom{e_t-e_{p+1}}{e_{t+1}-e_{p+1}}$ choices for T_{t+1}.

The number of quadruples $\{T_1, T_t, T_{t+1}, T_{p+1}\}$ such that T_{p+1} does not contain n and T_1 contains $(n+1)$ is hence given by:

$$\begin{aligned}
&\sum_c \binom{e_{p+1}-1}{c-1}\binom{n-e_{p+1}}{c}\binom{n-e_{p+1}-c}{e_1-1-e_{p+1}-c} \\
&[\binom{e_1-1-e_{p+1}}{e_t-1-e_{p+1}}\binom{e_t-1-e_{p+1}}{e_{t+1}-e_{p+1}}+\binom{e_1-1-e_{p+1}}{e_t-e_{p+1}}\binom{e_t-e_{p+1}}{e_{t+1}-e_{p+1}}] = \\
&\sum_c \binom{e_{p+1}-1}{c-1}\binom{e_1-e_{p+1}-1}{c}\binom{n-e_{p+1}}{e_1-e_{p+1}-1}\binom{e_1-1-e_{p+1}}{e_{t+1}-e_{p+1}}\binom{e_1-e_{t+1}}{e_t-e_{t+1}} =
\end{aligned}$$

$$\binom{e_1-2}{e_1-e_{p+1}-2}\binom{n-e_{p+1}}{e_1-e_{p+1}-1}\binom{e_1-1-e_{p+1}}{e_{t+1}-e_{p+1}}\binom{e_1-e_{t+1}}{e_t-e_{t+1}} \tag{15}$$

By summing up (14) and (15) and by applying lemma 1.3 we get

$$\binom{e_1-1}{e_{p+1}}\binom{n-e_{t+1}}{e_1-e_{t+1}-1}\binom{n-e_{p+1}}{e_{t+1}-e_{p+1}}\binom{e_1-e_{t+1}}{e_t-e_{t+1}}. \tag{16}$$

If T_1 does not contain $(n+1)$ (there are $\binom{n-c-e_{p+1}}{e_1-c-e_{p+1}}$ choices of such sets) then there are $\binom{e_1-e_{p+1}}{e_t-e_{p+1}}$ choices for T_t and $\binom{e_t-e_{p+1}}{e_{t+1}-e_{p+1}}$ choices

for T_{t+1}. The number of quadruples $\{T_1, T_t, T_{t+1}, T_{p+1}\}$ such that T_{p+1} does not contain n and T_1 does not contain $(n+1)$ is hence given by:

$$\sum_c \binom{e_{p+1}-1}{c-1}\binom{n-e_{p+1}}{c}\binom{n-e_{p+1}-c}{e_1-e_{p+1}-c}\binom{e_1-e_{p+1}}{e_t-e_{p+1}}\binom{e_t-e_{p+1}}{e_{t+1}-e_{p+1}} =$$

$$\sum_c \binom{e_{p+1}-1}{c-1}\binom{e_1-e_{p+1}}{c}\binom{n-e_{p+1}}{e_1-e_{p+1}}\binom{e_1-e_{t+1}}{e_t-e_{t+1}}\binom{e_1-e_{p+1}}{e_{t+1}-e_{p+1}} =$$

$$\binom{e_1-1}{e_{p+1}}\binom{n-e_{p+1}}{e_1-e_{p+1}}\binom{e_1-e_{t+1}}{e_t-e_{t+1}}\binom{e_1-e_{p+1}}{e_{t+1}-e_{p+1}} =$$

$$\binom{e_1-1}{e_{p+1}}\binom{n-e_{t+1}}{e_1-e_{t+1}}\binom{n-e_{p+1}}{e_{t+1}-e_{p+1}}\binom{e_1-e_{t+1}}{e_t-e_{t+1}} \tag{17}$$

By summing up (16) and (17) and by applying lemma 1.3 we get the desired (13).

We now prove (8). As for the case $p = 1$ (see lemma 1.4), the variety $Gr_{\mathbf{e}}(Reg_p^n(\lambda))$ equals the variety $Gr_{\mathbf{e}}(Reg_p^n(0))$ for every $\lambda \in k$. Indeed let us denote by $\circ$ and by $*$ respectively the action of $A = kQ_{p,1}$ on $Reg_p^n(\lambda)$ and on $Reg_p^n(0)$. We consider the automorphism ψ of the path algebra $kQ_{p,1}$ which sends ε_0 to $\lambda\pi + \varepsilon_0$ where $\pi := \varepsilon_1 \circ \cdots \circ \varepsilon_p$ is the longest path of $Q_{p,1}$. As vector spaces, $Reg_p^n(0)$ and $Reg_p^n(\lambda)$ are isomorphic. Then for every π in A and every m in $Reg_p^n(0)$, $\psi(\pi) * m = \pi \circ m$. This proves that they are right–equivalent in the sense of [13].

$$Reg_p^n(\lambda) := \begin{tikzcd} k^n \arrow[d, "="'] & \cdots \arrow[l, "="'] & k^n \arrow[l, "="'] \\ k^n & & k^n \arrow[ll, "J_n(\lambda)"] \arrow[u, "="'] \end{tikzcd}$$

We thus assume that $\lambda = 0$. In this case the representation $Reg_p^n(0)$ is an orientable string module by lemma 1.2 and we can therefore apply theorem 1. The Euler–Poincaré characteristic of $Gr_{\mathbf{e}}(Reg_p^n(0))$ is hence the number of $(p+1)$–tuples $\{T_1, \cdots, T_{p+1}\}$ of subsets $T_i \subset [1,n]$ of cardinality $|T_i| = e_i$ such that $T_{i+1} \subset T_i$ for $i = 1, \cdots, p$ and $J_n(0)(T_{p+1}) \subset T_1$ where $J_n(0) : [1,n] \to [1,n] \cup \{0\}$ is the map which sends k to $k-1$, $k \in [1,n]$. The choice of the couple $\{T_1, T_{p+1}\}$ determines the choice of $\binom{e_1-e_{p+1}}{e_2-e_{p+1}}$ choices for T_2. For every such choice there are $\binom{e_2-e_{p+1}}{e_3-e_{p+1}}$ choices for T_3, and so on. For every choice of $\{T_1, T_{p+1}\}$ there are hence $\chi_{\mathbf{e}}([1, p+1]) = \prod_{k=1}^{p-1} \binom{e_k-e_{p+1}}{e_{k+1}-e_{p+1}}$ choices for $\{T_2, \cdots, T_p\}$. The number of couples $\{T_1, T_{p+1}\}$ equals $\chi_{(e_1, e_{p+1})}(Reg_1^n(0))$. It remains to prove that:

$$\chi_{(e_1,e_2)}(Reg_1^n(0)) = \binom{e_1}{e_2}\binom{n-e_2}{e_1-e_2}$$

which has already been noticed in proposition 4.

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